Models of Set Theory II - Winter 2015/2016

Prof. Dr. Peter Koepke, Regula Krapf Problem sheet 2

Problem 7 (2 points)**.** Let κ be an infinite cardinal and $\mathbb{P}$ be a κ-closed partial order. Show that $\mathrm{FA}_\kappa(\mathbb{P})$ holds.

Problem 8 (2 points)**.** Prove that $\mathrm{FA}_{2^{\aleph_0}}(\mathrm{Fn}(\aleph_0, \aleph_0, \aleph_0))$ is false.

Problem 9 (6 points)**.** A set $X \subseteq \mathbb{R}$ is said to be *nowhere dense*, if there is a dense open set $U \subseteq \mathbb{R}$ which is disjoint from X. A set $X \subseteq \mathbb{R}$ is said to be *meager*, if it is a countable union of nowhere dense sets. Prove the following statements for $X \subseteq \mathbb{R}$:

(a) X is meager if and only if there is a sequence $\langle U_n \mid n \in \omega \rangle$ of dense open sets such that $X \cap \bigcap_{n \in \omega} U_n = \emptyset$.

(b) If MA holds then the intersection of less than $2^{\aleph_0}$-many dense open sets of reals is dense.

(c) Conclude that $\mathbb{R}$ is not meager.

Problem 10 (8 points)**.** Let $\kappa < 2^{\aleph_0}$ be an infinite cardinal and $\vec{X} = \langle X_i \mid i < \kappa \rangle$ be a sequence of nowhere dense sets. Let $\mathcal{I} = \{(p, q) \mid p, q \in \mathbb{Q}, p < q\}$ denote the set of open intervals in the real line with rational endpoints. Let $\mathbb{P} = \mathbb{P}_{\vec{X}}$ denote the forcing notion whose conditions are of the form $p = \langle r_p, s_p \rangle$ with the following properties:

- $r_p : \omega \to [I]^{<\omega}$ and $\{n \in \omega \mid r_p(n) \neq \emptyset\}$ is finite;
- $s_p : \omega \to [\kappa]^{<\omega}$ and $\{n \in \omega \mid s_p(n) \neq \emptyset\}$ is finite;
- for all $n \in \omega$ and for all $i \in s_p(n)$, $X_i \cap \bigcup r_p(n) = \emptyset$.

Furthermore, we define $p \leq_{\mathbb{P}} q$ if and only if for all $n \in \omega$, $r_p(n) \supseteq r_q(n)$ and $s_p(n) \supseteq s_q(n)$.

(a) Prove that $\mathbb{P}$ satisfies the ccc.

(b) For an interval $I \in \mathcal{I}$ and $n \in \omega$ let $D_{n,I} = \{p \in \mathbb{P} \mid \exists J \in r_p(n)(J \subseteq I)\}$. Show that $D_{n,I}$ is a dense subset of $\mathbb{P}$.

(c) Show that for every $i < \kappa$, the set $D_i = \{p \in \mathbb{P} \mid \exists n \in \omega(i \in s_p(n))\}$ is dense in $\mathbb{P}$.

(d) If MA holds then unions of less than $2^{\aleph_0}$-many meager sets are meager.

Hint for (d): Take a $\mathcal{D}$-generic filter for $\mathbb{P}_{\vec{X}}$ for some suitable $\vec{X}$ and $\mathcal{D}$ and show that the sets $U_n = \bigcup \bigcup \{r_p(n) \mid p \in G\}$ are dense open.

Please hand in your solutions on Monday, 16.11.2015 before the lecture.